Fixed-point iteration

1. Up to this point, we have discussed how if you are looking for a solution to $x = f(x)$, if you start with an initial guess to this solution x_0 and iterate $x_{k+1} \leftarrow f(x_k)$ **and** this sequence actually converges, then it converges to a solution to the equation above. The issue is that not all sequences converge. In other situations, it converges, but very slowly. We will explore some of these possibilities.

Which of the following converge, and how would you describe the convergence?

a. $x = \sin(x)$ with $x_0 = 1$
b. $x = \cos(x)$ with $x_0 = 1$
c. $x = \tan(x)$ with $x_0 = 0.0000001$

Answer: The first converges, but slowly. The second converges more quickly and alternates between a value that is greater than the solution, and a value that is less than the solution. The third diverges, but initially very slowly.

2. We have not yet described when fixed-point iteration converges, and when it does not. We have the following conditions. Suppose that x_{fixed} is a solution to $x = f(x)$. Then, we can say the following:

a. If $0 < f^{(1)}(x_{\text{fixed}}) < 1$, then we will see O($h$) convergence to the solution if we are sufficiently close to the solution initially, and the convergence will be monotonic; that is, the iterations will move in the same direction towards the solution. By O(h), we mean that if the error is h, then the error with the next iteration will be a constant times h.
b. If $f^{(1)}(x_{\text{fixed}}) = 0$, then we will see faster convergence to the solution.
c. If $-1 < f^{(1)}(x_{\text{fixed}}) < 0$, then we will also see O(h) convergence to the solution if we are sufficiently close to the solution initially, and the convergence will alternate between opposites sides of the solution, so if $x_k < x_{\text{fixed}}$ then $x_{k+1} > x_{\text{fixed}}$.
d. If $|f^{(1)}(x_{\text{fixed}})| > 1$, then regardless of how close we are to the solution (other than the solution itself), fixed-point iteration will diverge from this solution.
e. If $|f^{(1)}(x_{\text{fixed}})| = 1$, then we will see either very slow convergence or very slow divergence.

If $f(x) = cx(1 - x)$ is our function, where c is some real number, we can see many of these features. What are the solutions to $x = f(x)$ for this equation?

Answer: x = 0 and $x = \frac{c-1}{c}$.

3. What are the derivatives at each of these solutions given in Question 2 equal to?

Answer: At $x = 0$, the derivative is c while if $x = \frac{c-1}{c}$ then the derivative is $2 - c$.

For these next two questions, it may be best to write a program in C++ or MATLAB or a programming language of your choice to simulate this:

```
>> f = @(x, c)(c*x*(1 - x));

double f( double x, double c ) {
    return c*x*(1 - c);
}
```

4. If $|c| < 1$, to which solution will fixed-point iteration converge to for the function given in Question 2? Provide an argument to justify your answer, and demonstrate your answer with an example in C++ or MATLAB.

Answer: It will converge to $x = 0$ because the derivative there will be equal to c, which will be less than one in absolute value. If $-1 < c < 0$, the convergence will alternate between the sides of $x = 0$. If $0 < c < 1$, the convergence directly to $x = 0$. It will not converge at $x = \frac{c-1}{c}$ because the derivative is greater than one.

5. If $c = 1$, there is a double solution at $x = 0$. What is the rate of convergence to this solution? Provide an argument to justify your answer, and demonstrate your answer with an example in C++ or MATLAB.

Answer: It is very slow convergence, much less than $O(h)$.

6. If $1 < c < 2$, to which solution will fixed-point iteration converge to for the function given in Question 2? Provide an argument to justify your answer, and demonstrate your answer with an example in C++ or MATLAB.

Answer: It will converge to the second solution, $x = \frac{c-1}{c}$, and because the derivative is $2 - c$, this means the derivative will be positive, and therefore it will converge directly to the solution.

7. If $c = 2$, to which solution will fixed-point iteration converge to for the function given in Question 2? Provide an argument to justify your answer, and demonstrate your answer with an example in C++ or MATLAB.

Answer: It will converge to the solution $x = ½$, and it will be very fast convergence to that solution because ethe derivative there is equal to zero.

8. If $2 < c < 3$, to which solution will fixed-point iteration converge to for the function given in Question 2? Provide an argument to justify your answer, and demonstrate your answer with an example in C++ or MATLAB.

Answer: It will converge to the second solution, $x = \frac{c-1}{c}$, and because the derivative is $2 - c$, this means the derivative will be negative, and therefore the convergence will alternate between the sides of the solution.